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TITLE: FUN WITH SUPERSYMMETRIC QUANTUM MECHANICS

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FUN WITH SUPERSYMMETRIC QUANTUM MECHANICS

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Introduction

Recently there has been renewed interest in supersymmetric field theories[1] as a possible vehicle for making realistic phenomenological models of particle physics which do not suffer from the need to fine tune the bare masses in the boson sector of the theory. In ordinary theories the meson radiative correction are of order of Λ^2 where Λ is the cutoff, usually assumed to be the Grand unified scale ($\sim 10^{15}$ GeV). The fact that the generators of supersymmetry transformations convert bosons into fermions and that fermion masses can stay zero because of chiral symmetries allows a solution of the fine tuning problems.[2]

In a realistic model based on supersymmetry (SUSY), supersymmetry must be broken because of the lack of observed boson partners to the light leptons. One can prove that in weak coupling perturbation theory, if the classical (tree level) approximation to the theory is supersymmetric, then perturbative radiative corrections do not break supersymmetry. Thus, in making realistic model field theories, one must break SUSY at the tree level or nonperturbatively. It is thus important to have a nonperturbative method of determining if in a given theory supersymmetry is broken.

One reason for studying supersymmetric quantum mechanics is that there are a class of superpotentials $W(x)$ which behave at large x as x^α for which we know from general arguments whether SUSY is broken or unbroken. Thus one can use these superpotentials to test various ideas about how to see if supersymmetry is broken in an arbitrary model.

Recently,[3] Witten proposed a topological invariant, the Witten index Δ which counts the number of bosons minus the number of fermions having ground state energy zero. Since if supersymmetry is broken, the ground state energy cannot be zero, one expects if Δ is not zero, SUSY is preserved and the theory is not a good candidate for a realistic model. In this study we evaluate Δ for several examples, and show some unexpected peculiarities of the Witten index for certain choice of superpotentials $W(x)$.[4] In this survey we also discuss two other nonperturbative methods of studying supersymmetry breakdown. One involves relating supersymmetric quantum mechanics to a stochastic classical problem[5,6] and the other involves considering a discrete (but not supersymmetric) version of the theory and studying its behavior as one removes the lattice cuttoff.[7,8]

In this survey we review both the Hamiltonian and path integral approaches to supersymmetric quantum mechanics.[2,5,9] We then discuss the related path integrals for the Witten Index and for stochastic processes and show how they are indications for supersymmetry breakdown. We then discuss a system where the superpotenial $W(x)$ has assymetrical values at $\pm\infty$.[4,14] We find that in that case pairing is broken in that there is a mismatch between the fermionic and bosonic continuum density of states. Also, the Witten index is dependent on the regulation parameter β. We finally discuss nonperturbative strategies for studying supersymmetry breakdown based on introducing a lattice and studying the behavior of the ground state energy as the lattice cutoff is removed.[7,8]

II. Supersymmetric Quantum Mechanics and Spontaneous Supersymmetry Breaking in the Hamiltonian Formalism[2,5,9]

The sypersymmetry algebra in $D = 1$ is generated by the charges

$$\begin{aligned} Q^* &= (\hat{p} + i\, W(\hat{x}))\, \hat{\Psi}^* , \quad \{Q^*, Q^*\} = 0 \\ Q &= (\hat{p} - i\, W(\hat{x}))\, \hat{\Psi} , \quad \{Q, Q\} = 0 \end{aligned} \tag{1}$$

where

$$[\hat{p}, \hat{x}] = -i \quad , \quad \{\hat{\Psi}^*, \hat{\Psi}\} = 1 \quad .$$

One has that

$$\{Q^*, Q\} = 2H \tag{2}$$

so that

$$H = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}W^2(\hat{x}) - \frac{[\hat{\Psi}^*, \hat{\Psi}]}{2} W'(x) \tag{3}$$

$$[H, Q] = [H, Q^*] = 0 \quad .$$

One can also define a "fermion" number operator

$$(-1)^F = 1 - 2\,\Psi\,\Psi^* \tag{4}$$

which anti-commutes with Q, Q^*. One can realize the above algebra in one dimension by the following matrix representation for Ψ^* and Ψ.

$$\Psi^* = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \Psi = \sigma^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{5}$$

In this representation

$$(-1)^F = -\sigma_3 \quad , \quad \frac{[\Psi^*, \Psi]}{2} = -\tfrac{1}{2}\sigma_3 \tag{6}$$

and the Hamiltonian is diagonal

$$H = \tfrac{1}{2}[p^2 + W^2\ (x)] + \frac{\sigma_3}{2} W'(x) \tag{7}$$

or

$$H = \tfrac{1}{2} \left(\begin{array}{c|c} D_+ D_- & 0 \\ \hline 0 & D_- D_+ \end{array}\right) = \left(\begin{array}{c|c} H_- & \\ \hline & H_+ \end{array}\right) \tag{8}$$

$$D_\pm = \pm \frac{\partial}{\partial x} + W(x) = i \left\{ \begin{matrix} Q \\ -Q^* \end{matrix} \right\} \quad .$$

The eigenstates of H can be written as the vector

$$\begin{bmatrix} \psi_n^{(-)} \\ \psi_n^{(+)} \end{bmatrix} \tag{9}$$

where the ± correspods to $(-1)^F$ being ±1: i.e., the + corresponds to the bosonic states and the - to the fermionic states. From Eq. (8) and

$$H\,\Psi = E\,\Psi \tag{10}$$

we obtain

$$\begin{aligned} D_-\,\psi_n^{(-)} &= \sqrt{2E}\,\psi_n^{(+)} = -iQ^*\,\psi_n^{(-)} \\ D_+\,\psi_n^{(+)} &= \sqrt{2E}\,\psi_n^{(-)} = iQ\,\psi_n^{(+)} \end{aligned} \tag{11}$$

which shows how supersymmetry pairs the positive energy solutions.

Next we want to discuss supersymmetry breaking. For supersymmetry to be a good symmetry

$$Q|0\rangle = Q^*\,|0\rangle = 0|0\rangle \tag{12}$$

where $|0\rangle$ denotes the vacuum state. From (12) and (2) we find that for supersymmetry to be a good symmetry the ground state energy must be zero. From (11) we then see when supersymmetry is broken, $E_g \neq 0$, the ground state must be degenerate. One measure of supersymmetry breaking is the Witten index[3] $\Delta = \mathrm{Tr}(-1)^F$. Since the finite energy states are fermi-bose paired, this quantity measures $N_+(E=0) - N_-(E=0)$, the difference of zero energy boson and fermion states. Thus, if $\Delta \neq 0$, supersymmetry is unbroken. Usually $\mathrm{Tr}(-1)^F$ is ill defined so instead one considers

$$\Delta(\beta) = \mathrm{Tr}(-1)^F\,e^{-\beta H} = \mathrm{Tr}\left(e^{-\beta H_+} - e^{-\beta H_-}\right) . \tag{13}$$

Following Akhoury and Comtet[4] it is useful to introduce the heat kernels $K_\pm(x,y\ \beta)$ which satisfy

$$[\frac{d}{d\beta} - \frac{d^2}{dx^2} + W^2 \pm W'] K_\pm = 0 \tag{14}$$

$$K_\pm(x,y,\beta) = \langle y|e^{-\beta H}|x\rangle = \sum_n e^{-\beta E_n} \Psi_n^\pm(x) \Psi_n^\pm(y) \quad .$$

Using states normalized to one we obtain

$$\Delta(\beta) = \int dx[K_+ (x,x,\beta) - K_-(x,x,\beta)] \quad . \tag{15}$$

If there are also continuum states in the spectrum then one has

$$\Delta(\beta) = N_+(E = 0) - N_-(E = 0) + \int_{E_o}^{\infty} dE\ e^{-\beta E}[\rho_+(E) - \rho_-(E)] \tag{16}$$

where ρ (E) are the corresponding density of states. Thus, only if the density of states for bosons and fermions are different will $\Delta(\beta)$ depend on β. This will be important for our later discussion.

When the superpotential $W(x) \sim x^\alpha$ for large values of x, then it is easy to discuss supersymmetry breaking. For supersymmetry to be preserved one has

$$H\ \Psi_o = 0 \quad , \quad \Psi_o = \begin{bmatrix} \Psi_o^{(-)}\ (x) \\ \Psi_o^{(+)}\ (x) \end{bmatrix} \tag{17}$$

or from (11)

$$\begin{aligned} &(\frac{\partial}{\partial x} - W(x))\ \Psi_o^{(-)} = 0 \\ &(\frac{\partial}{\partial x} + W(x))\ \Psi_o^{(+)} = 0 \quad . \end{aligned} \tag{18}$$

Thus we obtain

$$\psi_0^{(-)} = A_1\, e^{\int W(x)dx} \quad , \quad \psi_0^{(+)} = A_2\, e^{-\int W(x)dx} \quad . \tag{19}$$

If $W(x) \sim bx^{\alpha}$ for large x then

$$\psi^{(-)} \sim A_1\, e^{\frac{bx^{\alpha+1}}{\alpha + 1}} \qquad\qquad \psi_2^{(+)} \sim A_2\, e^{-\frac{bx^{\alpha+1}}{\alpha + 1}} \quad . \tag{20}$$

We see when α is even there is no normalizable ground state with zero energy. When α is odd there is one normalizable ground state consisting of a single boson state.

$$\Psi_o = \begin{pmatrix} 0 \\ \psi_o^{(+)} \end{pmatrix} \quad , \quad \psi_o^{(+)} = A_2\, e^{-\int W dx} \tag{21}$$

such that $H\,\Psi_o = 0$. Thus we get the following picture of the spectrum when $W(x) \sim x^{\alpha}$ for large x: For α even SUSY is broken, $\Delta = 0$, $E_g > 0$ and all the eigenvalues have fermi-bose degeneracy. For α odd SUSY is a good symmetry, $\Delta = 1$, the ground state has $E = 0$ and is a bose state, all excited states have fermi-bose degeneracy.

III. Path Integral Formalism[5,10]

From the Hamiltonian (3) we have that

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}W^2(x) + i\,\Psi^* \partial_t \Psi - \frac{[\Psi^*, \Psi]}{2}\, W'(x) \quad . \tag{22}$$

Letting $t \to i\tau$ the Euclidean path integral is

$$Z[j, \eta, \eta^*] = \int[dx]\int[d\Psi]\int[d\Psi^*]\, e^{-S_E + \int_\tau (jx + \eta\Psi^* + \eta^*\Psi)d\tau} \tag{23}$$

where x is now a random variable and Ψ, Ψ^* are now elements of a Grassman algebra

$$\{\Psi^*,\Psi\} = \{\Psi,\Psi\} = \{\Psi^*,\Psi^*\} = 0$$

$$S_E = \int_o^\tau L_E \, d\tau$$

$$L_E = \tfrac{1}{2}(\partial_\tau x)^2 + \frac{W^2}{2}(x) - \Psi^*[\partial_t - W'(x)]\Psi \quad . \tag{24}$$

For the path integral $Z[j]$ the boundary conditions on the fermion fields are $\Psi(0) = -\Psi(\tau)$ We next want to integrate out the fermi field. We obtain

$$Z[j,\eta,\eta^*] = \int[Dx] \, \mathrm{Det}[\partial_\tau - W'(x)] \, \exp[-\int_o^T (\dot{x}^2/2 + \tfrac{1}{2}W^2) \, d\tau] \tag{25}$$

$$\exp\,[\int_o^T \eta^* \, [\partial_\tau - W'(x)]^{-1} \, \eta d\tau$$

The determinant has been evaluated by Gildener and Patrascioiu.[10]

$$\mathrm{Det}[\partial_\tau - W'] = \prod_m \lambda_m \text{ where } \lambda_m \text{ satisfy}$$

$$[\partial_\tau - W'(x)] \, \Psi_m = \lambda_m \, \Psi_m$$

$$[-\partial_\tau - W'(x)] \, \Psi_m^* = \lambda_m \, \Psi_m \quad . \tag{26}$$

Thus

$$\Psi_m = C_m \exp \int_{\tau_o}^{\tau} d\tau' [\lambda_m + W'(x(\tau'))] \quad . \tag{27}$$

Imposing antiperiodic boundary conditions $\Psi_m(T) = -\Psi_m(0)$ yields

$$\lambda_m = \frac{i(2m+1)\pi}{T} - \frac{1}{T}\int_0^T d\tau\, W'(x) \quad . \tag{29}$$

Thus

$$\prod_m \lambda_m(g)/\lambda_m(0) = \cosh \int_0^T d\tau \frac{W'(x)}{2} \quad . \tag{30}$$

Thus

$$Z = Z_- + Z_+$$

$$Z_\pm = \int [dx] \exp[-S_E^\pm] \tag{31}$$

$$S_E^\pm = \int_0^T d\tau \left(\frac{\dot{x}^2}{2} + \frac{W^2}{2}(x) \pm \frac{W'(x)}{2}\right) \quad .$$

Thus as expected, from (7)

$$Z = \mathrm{Tr}\, e^{-HT} = \mathrm{Tr}\, e^{-H_+T} + \mathrm{Tr}\, e^{-H_-T} \tag{32}$$

$$= Z_+ + Z_-$$

$$= \Sigma\, e^{-E_n^{(+)}T} + \Sigma\, e^{-E_n^{(-)}T} = \Sigma\, e^{-E_nT}$$

$$\{E_n\} = \{E_n^{(+)}, E_n^{(-)}\} \quad .$$

If SUSY is unbroken, then $E_o^{(+)} = 0$, $E_o^{(-)} > 0$ and to satisfy the degeneracy condition for $n \geq 1$

$$E_n^{(+)} = E_{n-1}^{(-)} \quad . \tag{33}$$

If SUSY is broken then $E_n^{(+)} = E_n^{(-)} > 0$ for all n, (n = 0, 1, 2....).

Now from this last discussion we expect that the regulated Witten index also has a simple path integral form. Since from (13)

$$\Delta(\beta) = \mathrm{Tr}\ (-1)^F\ e^{-\beta H} = \mathrm{Tr}\left(e^{-\beta H_+} - e^{-\beta H_-}\right) \quad , \tag{34}$$

we expect

$$\Delta(\beta) = Z_+ - Z_- \tag{35}$$

where in $S^{\pm}$ we integrate from zero to β instead of T. This is indeed so. To evaluate Tr $(-1)^F\ e^{-\beta H}$ we need to insert eigenstates of $|x\rangle$ and $|\Psi\rangle$ into the trace. Since $(-1)^F\ |b\rangle = |b\rangle$ and $(-1)^F\ |f\rangle = -|f\rangle$ one obtains for $\Delta(\beta)$

$$\Delta(\beta) = \int [dx] \int [d\Psi] \int [d\Psi^*]\ e^{\int_o^\beta L_E(x,\Psi,\Psi^*)d\tau} \tag{36}$$

but now because of the $(-1)^F$

$$x(0) = x(\beta) \quad ; \quad \Psi(0) = \Psi(\beta) \quad , \quad \Psi^*(0) = \Psi^*(\beta) \tag{37}$$

so we have a determinant with periodic boundary conditions. Imposing $\Psi_m(T) = \Psi_m(0)$ in (17) yields

$$\lambda m = i\,\frac{2m}{T}\,\pi - \frac{1}{T}\int_o^T d\tau\ W'(x) \tag{38}$$

so that

$$\Pi\ \lambda_m(g)/\lambda_m(0) = \sinh \int_o^T d\tau \quad \frac{W'(x)}{2} \tag{39}$$

and we obtain

$$\Delta(\beta) = Z_+ - Z_- = \Sigma\, e^{-E_n^{(+)}\beta} - \Sigma\, e^{-E_n^{(-)}\beta} \quad . \tag{40}$$

When SUSY is unbroken then $E_{n+1}^{(+)} = E_n^{(-)}$ and $E_o^{(+)} = 0$. In that case, $\Delta = 1$, independent of β. When SUSY is broken, then $E_n^{(-)} = E_n^{(+)} > 0$ for all n and $\Delta = 0$, independent of β. There is also a related path integral connected with the classical stochastic processes defined by the Langevin equation[11,12,13,5]

$$\dot{x}(\tau) = W(x(\tau)) + f(\tau) \tag{41}$$

where $f(\tau)$ is a random stirring force having Gaussian statistics.

$$P[f] = N \exp[-\tfrac{1}{2} \int_{to}^{T} dt\, \frac{f^2(t)}{F_o}] \tag{42}$$

so

$$\begin{aligned} &\int Df\; P[f] = 1 \\ &\int Df\; P[f]\; f(\tau) = 0 \\ &\int Df\; P[f]\; f(\tau)\; f(\tau') = F_o\; \delta(\tau - \tau') \quad . \end{aligned} \tag{43}$$

To determine correlation functions in x resulting from the statistics of the forcing term one has

$$P[f]\; Df = P[\dot{x} - W(x)]\; \det \left|\frac{\delta f}{\delta x}\right|\; Dx \quad . \tag{44}$$

Now we have

$$\frac{\delta f(\tau)}{\delta x_{(\tau')}} = [\frac{d}{d\tau} - W'(x(\tau))]\; \delta(\tau - \tau') = G^{-1}(\tau - \tau') \tag{45}$$

$$= \frac{d}{d\tau}\, [\delta(\tau - \tau') - \theta(\tau - \tau')W'(x)] = \frac{d}{d\tau}\, K(\tau, \tau') \quad .$$

Where we have used the fact that this is a causal system with forward propagation in time

$$G_o(\tau - \tau') = \Theta(\tau - \tau') \quad , \quad \frac{dG_o}{d\tau} = \delta(\tau - \tau') \quad . \tag{46}$$

Using this boundary condition only the first term in the expansion of the Tr ℓn survives, and apar from overall normalization

$$\det \left|\frac{\delta f}{\delta x}\right| = e^{\mathrm{Tr}(\ln K)} = e^{\frac{1}{2}\int W'(x)d\tau} \quad . \tag{47}$$

Parisi and Sourlas introduced Grassman variables to represent the determinant

$$I = \int [dx][d\Psi] \exp(-\tfrac{1}{2} \int [\dot{x}^2 + W^2(X) + \bar{\Psi}(t)[\frac{\partial}{\partial\tau} - W'] \Psi(t)]) \quad . \tag{48}$$

However antiperiodic b.c. on Ψ give $Z_+ - Z_-$ and period b.c. on Ψ give $Z_+ + Z_-$. Forward propagation boundary condition leads to

$$1 = Z_{stoch} = \int P[f]Df = \int dx\, e^{-\frac{1}{2}\int_0^\tau [\dot{x}^2 + W^2(x) - W'(x)]dt} \tag{49}$$

$$= Z_+ = \Sigma\, e^{-E_n^{(+)}\tau} \tag{50}$$

with the identification of $F_o = \frac{1}{2}$ and we have used $\int \dot{x}W(x)d\tau = \int \partial_\tau F(x(\tau)) d\tau = 0$ for a polynomial $W(x)$ and periodic $x(\tau)$. In deriving (49) we also assumed that Eq. (41) had only one solution for a given f.

When supersymmetry is broken and $T \to \infty$, $E_o^{(+)} > 0$ and one cannot satisfy $Z_+ = 1$. Thus the stochastic problem must become non-invertible.[6] In fact, the equation

$$\begin{aligned} \dot{x} &= -g\, x^n \\ x(t_o) &= x_o \end{aligned} \tag{51}$$

has solution[5]

$$x = [(n - 1)(gt + c)]^{\frac{1}{(1 - n)}}$$

where

$$c = \frac{x_o^{1-n}}{(n-1)} - gt_o \quad .$$

This solution can blow up for $t > t_o$ when n is even for certain x_o but not when n is odd. Furthermore, one can define a classical probability

$$P_{c\ell}(y,\tau) \equiv \langle \delta(y - x(\tau)) \rangle_f = \int Df\ P(f)\ \delta(y - x(\tau)) \tag{53}$$

such that

$$\int dy\ y^n\ P_{c\ell}(y) \equiv \int Df\ P[f]\ (x(\tau))^n = \langle x^n\ (\tau)\rangle \quad . \tag{54}$$

$P_{c\ell}$ satisfies Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \tfrac{1}{2}Fo\ \frac{\partial^2}{\partial y^2} P + \frac{\partial}{\partial y} (W(y)\ P\ (y,t)) \quad . \tag{55}$$

for an equilibrium distribution to exist at long times t one has

$$P\ \ (y,t) \to \bar{P}(y) \tag{56}$$

and $\int \bar{P}(y)dy = 1$.

Setting $\frac{\partial P}{\partial t} = 0$, we obtain

$$\bar{P}(y) = N \quad e^{-2/\hbar \int W(y)dy} \tag{57}$$

$$\equiv |\Psi_o(y)|^2$$

since $\Psi_o(y) = A\ e^{-\int W(y)dy}$ from (21) and $F_o = \hbar$. Thus, only when SUSY is unbroken ($E_o = 0$) does an equilibrium distribution exist for the classical stochastic system.

IV. SUSY Without Pairing[4,14]

When we allow W(x) to exhibit solitonic behavior so that

$$\begin{aligned} &\Phi(x)\to\Phi_+ \text{ at } x = +\infty \\ &\Phi(x)\to\Phi_- \text{ at } x = -\infty \end{aligned} \tag{58}$$

then the spectrum has a continuum as well as a finite number of bound states. In these cases supersymmetry actually forces an asymmetry in the density of states $\rho^+(E) \neq \rho^-(E)$ and therefore also the Witten index becomes β dependent. If we choose W(X) = tanh x then the Hamiltonian $H_\pm$ are

$$\begin{aligned} H_+ &= -\frac{d^2}{dx^2} + 1 - 2\ \mathrm{sech}^2 x \\ H_- &= -\frac{d^2}{dx^2} + 1 \quad . \end{aligned} \tag{59}$$

For H_- there are only scattering states $\Psi^{(-)}$ which are defined by their asymptotic behavior

$$\begin{aligned} &f_-(x,k) \sim e^{+ikx} \quad , \quad x\to\infty \\ &g_-(x,k) \sim e^{-ikx} \quad\quad x\to-\infty \\ &E_-(k) = (1 + k^2) \end{aligned} \tag{60}$$

whereas for H_+ there is one bound state

$$\psi_o^{(+)} = \text{sech } x \quad , \quad E_o^{(+)} = 0 \tag{61}$$

as well as the scattering states obtained by the supersymmetry operation (11)

$$\sqrt{2(1 + k^2)}\ \psi^{(+)} = (-\frac{\partial}{\partial x} + \tanh x)\psi^{(-)} \tag{62}$$

$$\therefore \sqrt{2(1 + k^2)}\ f_+ \sim (-ikx + \tanh x)e^{ikx} \qquad x \to +\infty$$

$$\therefore \sqrt{2(1 + k^2)}\ g_+ \sim (+ikx + \tanh x)e^{-ikx} \qquad x \to -\infty$$

$$E_+(k) = k^2 + 1 \quad .$$

Now the boundary conditions requires that $\psi^{(+)}(k)$ and $\psi^{(-)}(k)$ have different phase shifts making it impossible for the density of states $\rho^+(E)$ to be equal to $\rho^{(-)}(E)$. In fact, using the boundary condition that $W(x) \to \pm 1$ at $\pm\infty$ and the supersymmetry relation between $\psi^{(+)}$ and $\psi^{(-)}$ one can obtain a relationship between the Jost functions and derive

$$\rho^+(E) - \rho^-(E) = -\frac{1}{\pi E(E^2 - 1)^{\frac{1}{2}}} \quad . \tag{63}$$

We have from (16)

$$\begin{aligned} \Delta(\beta) &= 1 + \int_1^\infty dE e^{-\beta E}(\rho^+(E) - \rho^-(E)) \\ &= 1 - \frac{1}{\sqrt{\pi}}\,\Gamma(\tfrac{1}{2},\beta) \quad . \end{aligned} \tag{64}$$

This explicitly shows the β dependence of $\Delta(\beta)$. Thus, in this case the index is not an integer and is β dependent.

IV. Nonperturbative Strategies for Studying SUSY Breaking

Two methods for exploring field theory in the strong coupling (nonperturbative) domain exist. One is strong coupling expansions of lattice versions of the field theory[15] (on the lattice a kinetic energy or strong coupling expansion is a non-singular expansion). The other is Monte Carlo evaluation of the latticized Path Integral.[16] Here we ignore the problem of trying to make a supersymmetric lattice theory.[17] Instead we introduce a time lattice with lattice spacing a. The lattice breaks all, or part of SUSY algebra. Clearly the ground state energy of the lattice Hamiltonian will not be zero. However we can ask what happens to $E_o(a)$ as the lattice spacing a goes to zero. Although it might be difficult in an approximation scheme to see if $E_o = 0$, because we have an extra parameter, the lattice spacing one can ask whether

$$E_o(a) \sim a^{\gamma} \tag{65}$$

as $a \to 0$. Here γ is a critical exponent. If we find that $\gamma > 0$ within the accuracy of our calculational scheme, then one could say with confidence that the continuum theory was supersymmetric. If $\gamma \to 0$ and $E_o(a) \to$ finite constant as $a \to 0$ then we expect the continuum theory to break supersymmetry. So

$$\begin{aligned} &\gamma > 0 \leftrightarrow E_o = 0 \qquad &&\text{(SUSY)} \\ &\gamma = 0 \leftrightarrow E_o > 0 \qquad &&\text{(Broken SUSY)} \end{aligned} \tag{66}$$

If we perform a Monte Carlo calculation at two decreasing small lattice spacings

$$\gamma = \ln \frac{E_o(a')}{E_o(a)} \Big/ \ln (a'/a) \tag{67}$$

in the limit a' and $a \to 0$.

An analytic procedure is the strong coupling expansion of the Langevin equation.[7] For example, for $W(x) = gx^3$, we have on the lattice Eq. (41) becomes

$$\varepsilon(x_n - x_{n-1}) + gx_n^3 = f_n \tag{68}$$

where $\varepsilon = \frac{1}{a}$.

One then calculates

$$x_n = \Sigma \varepsilon^m x_n^{(m)}$$

and finds

$$x_n = \frac{f_n^{1/3}}{g^{1/3}} + \frac{\varepsilon}{3g^{2/3}} (f_{n-1}^{1/3} f_n^{-2/3} - f_n^{-1/3}) + 0(\varepsilon^2) \quad ..... \tag{69}$$

The ground state energy for a quantum mechanical system with $V(x) = \frac{g^2x^6}{2} - \frac{3}{2}gx^2$ is given by the Virial Theorem to be

$$E_o = 2g^2 \langle x^6 \rangle - 3g\langle x^2 \rangle \quad . \tag{70}$$

To perform the average over the noise we use

$$P[f] = \prod_i e^{-af^2(i)/2} \sqrt{\frac{a}{2\pi}} \quad . \tag{71}$$

The integrals over the noise are just Λ functions. We find at the m'th order in ε

$$E_o = \sqrt{g}\, z \sum_{n=0}^{m} C_n z^n \tag{72}$$

where

$$z^3 = \left(\frac{2}{a\sqrt{g}}\right)$$

is a dimensionless correlation length. One needs to extrapolate this finite series from small z to infinite z. To do this one uses Pade approximants. Assuming

$$E_o \sim z^{\alpha}$$

$$\alpha = \lim_{z\to\infty} z \frac{E_o'(z)}{F_o(z)} \tag{73}$$

we obtain a sequence of approximants

$$\alpha = +1,\ +.4766,\ -3.507,\ -3.4903,\ -3.4997,\ -3.4763. \tag{74}$$

Thus we estimate since $\gamma = -\alpha/3$

$$E_o \sim a^{1.16} \quad . \tag{75}$$

This agrees with our knowledge that SUSY is good when $W = gx^3$. It also gives credence to the idea that one can naively put SUSY on a lattice and have it restored in the continuum.

We can also check this method when SUSY is broken. If we choose $W(x) = gx^2/2$, we can write[8] a strong coupling series of E_o using path integral techniques:

$$Z[J] = \exp[\tfrac{1}{2} \int dt \int dt' \frac{\delta}{\delta J(t)} G_o^{-1} (t,t') \frac{\delta}{\delta J(t')}]\ Z_o\ [J] \tag{76}$$

$$Z_o\ [J] = \int D\, x \exp(- \int dt\ [\tfrac{1}{2}W^2 - \frac{W'(x)}{2} - J(t)\ x(t)]) \quad .$$

One the lattice $t = na$, $W(x) = W(x_n)$

$$Z_o\ [J] = \Pi \frac{F(J_n)}{F(0)} = N \exp \sum_n \ln\ [F(J_n)/F(0)] \tag{77}$$

$$F(y) = \int_{-\infty}^{\infty} dt\ \exp\ (-\frac{1}{2}\ aW^2 + \frac{1}{2}\ aW' + ayt)$$

$$= \Sigma\ A_n\ y^n/n!$$

$$A_n = \int dt\ t^n \exp[-ag\ t^4/8 + ag\ t/2]$$

$$\frac{A_n}{A_o} = (\sqrt{3a}\ \varepsilon)^n\ \Sigma a_{nm}\ \varepsilon^{2m}$$

$$\varepsilon = g^{-1/3}\ (3a)^{-1/2} \quad .$$

The inverse propagator $\partial_t^2\ \delta(t - t')$ becomes well behaved on the lattice

$$G_o^{-1}\ (n,m) = \frac{1}{a^3}\ (\delta_{n,\ m+1} + \delta_{m,\ n+1} - 2\delta_{nm}) \quad . \tag{78}$$

The expansion in powers of G_o^{-1} gives a series in ε. We obtain

$$\begin{aligned} E_o &= \frac{3g^2}{8} \langle x^4 \rangle - \frac{3}{4}\ g\langle x \rangle \\ &= g^{2/3}\ \Sigma\ C_n\ \varepsilon^{2n} \end{aligned} \tag{79}$$

Analyzing this series we find

$$E_o \neq 0 \text{ and } \gamma \lesssim 0 \quad .$$

Showing no evidence for SUSY as expected.

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